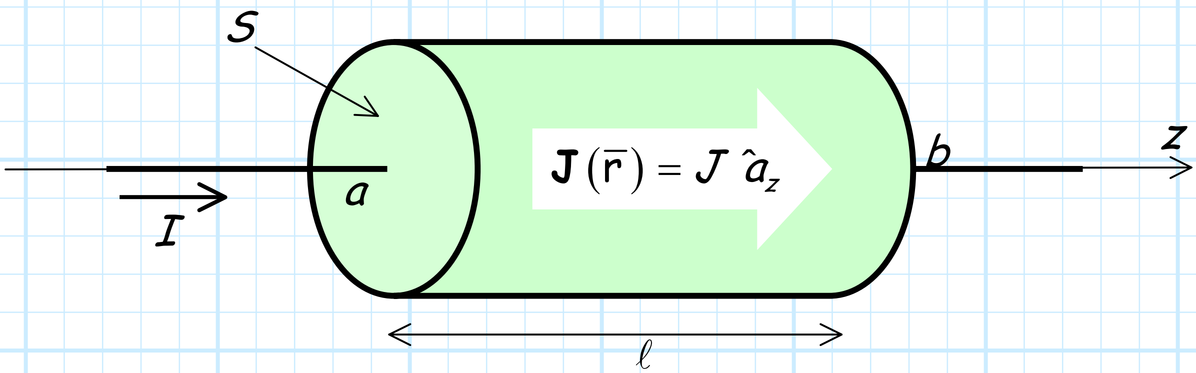
Resistors

Consider a **uniform** cylinder of material with mediocre to poor to pathetic **conductivity** $\sigma(\bar{\mathbf{r}}) = \sigma$.



This cylinder is centered on the *z*-axis, and has **length** ℓ. The **surface area** of the ends of the cylinder is *S*.

Say the cylinder has **current** *I* flowing into it (and thus out of it), producing a current **density** $\mathbf{J}(\bar{\mathbf{r}})$.

By the way, this cylinder is commonly referred to as a **resistor**!

Q: *What is its* ***resistance*** *R of this resistor, given length* ℓ*, cross-section area S, and conductivity* σ*?*

A: Let's first begin with the circuit form of Ohm's Law:

$$R = \frac{V}{I}$$

where V is the potential difference between the two ends of the resistor (i.e., the voltage across the resistor), and I is the current through the resistor.

From **electromagnetics**, we know that the potential difference V is:

$$V = V_{ab} = \int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

and the current I is:

$$I = \iint_S \mathbf{J}(\bar{r}) \cdot \overline{ds}$$

Thus, we can **combine** these expressions and find resistance R, expressed in terms of electric field $\mathbf{E}(\bar{r})$ within the resistor, and the current density $\mathbf{J}(\bar{r})$ within the resistor:

$$R = \frac{V}{I} = \frac{\int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell}}{\iint_S \mathbf{J}(\bar{r}) \cdot \overline{ds}}$$

Lets evaluate **each integral** in this expression to determine the resistance R of the device described earlier!

1) The voltage V is the potential difference V_{ab} between point a and point b:

$$V = V_{ab} = \int_a^b \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell}$$

Q: *But, what is the electric field $\mathbf{E}(\bar{\mathbf{r}})$?*

A: The electric field within the resistor can be determined from **Ohm's Law**:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{\mathbf{J}(\bar{\mathbf{r}})}{\sigma(\bar{\mathbf{r}})}$$

We can assume that the **current density** is approximately **constant** across the cross section of the cylinder:

$$\mathbf{J}(\bar{\mathbf{r}}) = J\,\hat{a}_z$$

Likewise, we know that the conductivity of the resistor material is a **constant**:

$$\sigma(\bar{\mathbf{r}}) = \sigma$$

As a result, the electric field **within** the resistor is:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{\mathbf{J}(\bar{\mathbf{r}})}{\sigma(\bar{\mathbf{r}})} = \frac{J}{\sigma}\,\hat{a}_z$$

Therefore, **integrating** in a straight line along the *z*-axis from point *a* to point *b*, we find the potential difference *V* to be:

$$
\begin{aligned}
V &= \int_a^b \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell} \\
&= \frac{1}{\sigma} \int_{z_a}^{z_b} J \, \hat{a}_z \cdot \hat{a}_z \, dz \\
&= \frac{J}{\sigma} \int_{z_a}^{z_b} dz \\
&= \frac{J\ell}{\sigma}
\end{aligned}
$$

2) We likewise know that the current *I* through the resistor is found by evaluating the **surface integral**:

$$
\begin{aligned}
I &= \iint_S \mathbf{J}(\bar{\mathbf{r}}) \cdot \overline{ds} \\
&= \iint_S J \, \hat{a}_z \cdot \hat{a}_z \, ds_z \\
&= J \iint_S ds_z \\
&= J \, S
\end{aligned}
$$

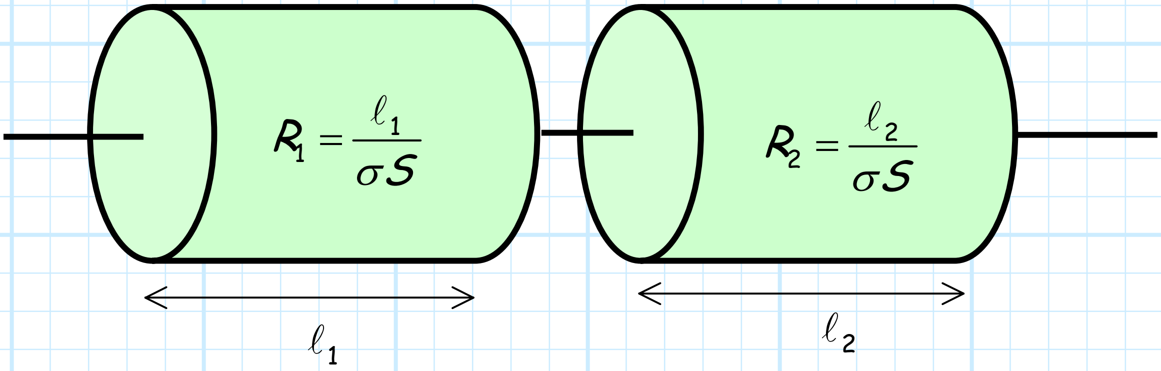
Therefore, the resistance *R* of **this particular** resistor is:

$$
\begin{aligned}
R &= \frac{V}{I} \\
&= \left(\frac{J\,\ell}{\sigma}\right)\left(\frac{1}{J\,S}\right) \\
&= \frac{\ell}{\sigma S}
\end{aligned}
$$

An interesting result! Consider a resistor as sort of a "clogged pipe". **Increasing** the cross-sectional area S makes the pipe bigger, allowing for **more current** flow. In other words, the resistance of the pipe decreases, as predicted by the above equation.

Likewise, increasing the **length** ℓ simply increases the length of the "clog". The current encounters resistance for a longer distance, thus the value of R increases with increasing length ℓ. Again, this behavior is predicted by the equation shown above.

For **example**, consider the case where we add two resistors together:

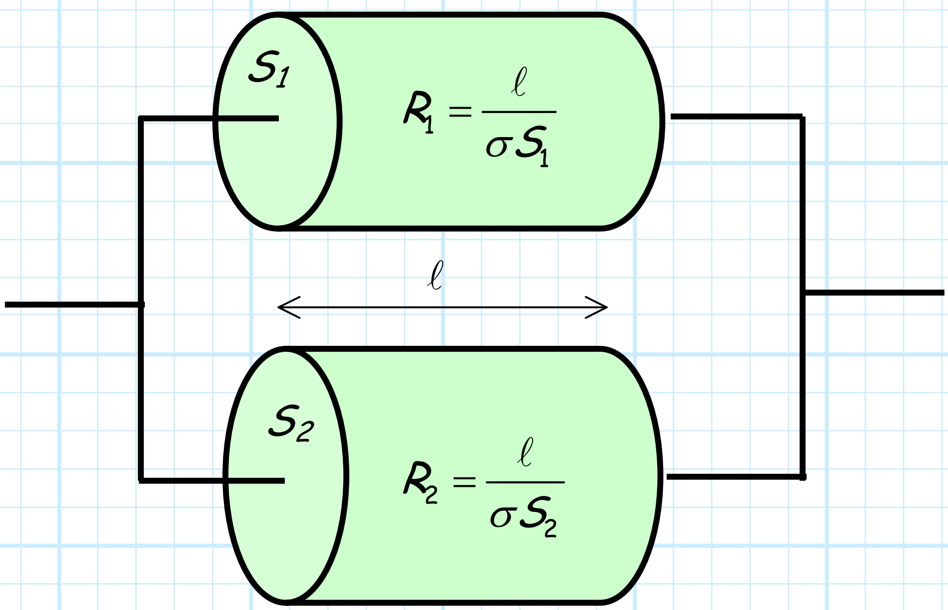


We can view this case as a single resistor with a length $\ell_1 + \ell_2$, resulting in a total resistance of:

$$\begin{aligned} R_{total} &= \frac{\ell_1 + \ell_2}{\sigma S} \\ &= \frac{\ell_1}{\sigma S} + \frac{\ell_2}{\sigma S} \\ &= R_1 + R_2 \end{aligned}$$

But, this result is not the **least bit** surprising, as the two resistors are connected in **series**!

Now let's consider the case where two resistors are connected in a different manner:



We can view this as a single resistor with a total cross sectional area of $S_1 + S_2$. Thus, its total resistance is:

$$\begin{aligned} R_{total} &= \frac{\ell}{\sigma\left(S_1 + S_2\right)} \\ &= \left[\frac{\sigma\left(S_1 + S_2\right)}{\ell}\right]^{-1} \\ &= \left[\frac{\sigma S_1}{\ell} + \frac{\sigma S_2}{\ell}\right]^{-1} \\ &= \left[\frac{1}{R_1} + \frac{1}{R_2}\right]^{-1} \end{aligned}$$

Again, this should be no surprise, as these two resistors are connected in **parallel**.

IMPORTANT NOTE: The result $R = \ell / \sigma S$ is valid **only** for the resistor described in this handout. Most importantly, it is valid only for a resistor whose conductivity is a **constant** ($\sigma(\bar{r}) = \sigma$).

If the conductivity is **not** a constant, then we **must** evaluate the potential difference across the resistor with the more **general** expression:

$$\begin{aligned} V_{ab} &= \int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} \\ &= \int_a^b \frac{\mathbf{J}(\bar{r})}{\sigma(\bar{r})} \cdot \overline{d\ell} \end{aligned}$$